GCE: 503, Analysis and measure theory

August 2016

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Exercise 1:

Suppose that u is a real-valued function defined on $[0, 1]$, that $u \geq 0$ and that $u \in L^1([0, 1])$. Define $E_n := \{x \in [0, 1] : n - 1 \leq u(x) \leq n\}$ for each positive integer n. Show that

$$\sum_{n=1}^{+\infty} n\,|E_n| < +\infty \;.$$

Exercise 2:

Show that a subset E of a metric space X is open if and only if there is a continuous real-valued function f on X such that $E = \{x \in X : f(x) > 0\}$.

Exercise 3:

Consider the sequence of functions $\{f_n\}$ defined on the non-negative reals: $[0, +\infty)$ where $f_n(x) = 2nxe^{-nx^2}$. Let g be a continuous and bounded function on $[0, +\infty)$ valued in $\mathbb{R}$.

1. Find, with proof,

$$\lim_{n\to\infty} \int_0^{\infty} f_n(t)g(t)dt$$

2. Define for x in $[0, +\infty)$,

$$g_n(x) = \int_0^{\infty} f_n(t)g(x + t)dt.$$

   Assuming g is zero outside the interval $[0, M]$, where $M > 0$, does the sequence g_n have a limit in $L^1([0, +\infty))$?

3. If h is in $L^1([0, +\infty))$, define for x in $[0, +\infty)$,

$$h_n(x) = \int_0^{\infty} f_n(t)h(x + t)dt.$$

   Show that h_n is measurable on $[0, +\infty)$ and is in $L^1([0, +\infty))$.

4. Find, if it exists, with proof, the limit of h_n in $L^1([0, +\infty))$.

Exercise 4:

Show that a set $E \subset \mathbb{R}$ is Lebesgue measurable if and only if $E = H \cup Z$ where H is a countable union of closed sets and Z has measure zero. You may use the following poperty: for any Lebesgue measurable subset A of $\mathbb{R}$ and any $\epsilon > 0$, there is a closed subset F of $\mathbb{R}$ such that $F \subset A$ and the measure of $A \setminus F$ is less than ϵ.

Exercise 5:

Give an example of a sequence f_n in $L^1(0,1)$ such that $f_n \to 0$ in $L^1(0,1)$ but f_n does not converge to zero almost everywhere.